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THE ASYMPTOTIC SOLUTION OF THE PROBLEM OF A THIN THREE-DIMENSIONAL BODY ENTERING A COMPRESSIBLE FLUID†

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A uniformly applicable solution is constructed in the neighbourhood of the leading edges of a thin three-dimensional body penetrating into a compressible fluid. Examples of such solutions are given for thin cyclically-symmetric bodies with plane facets for various entry conditions. Copyright © 1996 Elsevier Science Ltd.

Solutions of linear problems involving the entry of thin bodies into fluids exhibit a characteristic feature, in that certain physical quantities relating to the disturbed flow diverge both in the neighbourhood of the curves in which the body intersects the free surface of the fluid and in the neighbourhood of the sharp nose of the body (in the two-dimensional and axisymmetric problems [1–3]) or of the sharp leading edges [4, 5] submerged in the fluid. Uniformly valid solutions in the neighbourhood of the apex of a wedge or a cone in an acoustic setting were obtained in [6, 7].

The domain of inhomogeneity of the exterior (linear) solution in the three-dimensional problem in a "tube" sheathing the neighbourhood of the sharp leading edge, with small transverse scales. The inner problem reduces to solving a two-dimensional Laplace equation for the inner potential in a plane normal to the leading edge at some point of the latter, on the assumption that the Riemann–Hilbert condition holds on the faces of the "wedge" formed by the edge in the neighbourhood of the point. Examples are presented of uniformly valid solutions for various conditions of entry of thin conical bodies with a rhomboid transverse profile, moving at a constant velocity normal to the free surface of the fluid, and formulae are given for the pressure at the leading edges. Attention is paid to the special features of the construction of a uniformly valid solution for the entry of a thin cyclically-symmetric body (CSB)—a sheaf of an integral number of identical thin three-dimensional bodies (cycles) with sharp leading edges, symmetrically arranged round a longitudinal axis. The pressure at the edge of the CSB in domains of mutual influence of the cycles is determined by the sum of the pressure at the edge, evaluated in the main problem of the entry of a single cycle by using a uniformly valid solution and the non-linear Cauchy–Lagrange integral, and the pressure perturbation introduced at the point in question by the other cycles and evaluated on the basis of the linear solution.

1. FORMULATION OF THE PROBLEM. CONSTRUCTION OF A UNIFORMLY VALID SOLUTION

Consider a thin three-dimensional body penetrating a fluid-filled half-plane, at a velocity $v_0(t)$ whose direction is assumed (for simplicity) to be that of the inner normal to the free surface of the fluid. Let us assume that the body shape and the entry conditions guarantee that the flow around the body is not detached, and that a solution is known of the corresponding linear problem for the potential of the perturbed fluid flow; we shall henceforth refer to this solution as the outer solution $\varphi_e(x_1, y_1, z_1, t)$, where x_1, y_1, z_1 is an absolute Cartesian system of coordinates attached to the free surface of the undisturbed fluid, with x_1 axis in the direction of the body's velocity $v_0(t)$.

We will write the wave equation, the Cauchy–Lagrange integral and the boundary condition on the body; these will be needed to construct a uniformly valid solution in the neighbourhood of the leading edges of the body

$$\frac{\partial^2\varphi}{\partial t^2} = c^2\Delta\varphi \tag{1.1}$$

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$$\frac{p-p_0}{\rho_0}+\frac{\partial\varphi}{\partial t}+\frac{1}{2}(\nabla\varphi)^2=0 \tag{1.2}$$

$$\frac{\partial\varphi}{\partial n_1}=(\mathbf{n}_1,\mathbf{v}_0(t)) \tag{1.3}$$

where c, p_0 and ρ_0 are the speed of sound, the pressure and density of the undisturbed fluid, respectively, $\mathbf{n}_1$ is the unit vector of the outward normal to the body surface. In thin bodies, $(\mathbf{n}_1, \mathbf{v}_0(t)) = O(\varepsilon)$, where $\varepsilon \ll 1$ is a small parameter characterizing the relative thickness of the body. The potential of perturbed fluid flow is also of the order of ε, and its principal term φ_c is a solution of a linear initial-boundary-value problem.

Solutions of linear three-dimensional problems of thin bodies entering a fluid [4, 5], for velocity and pressure determined from the linearized Cauchy–Lagrange integral (1.2), have singularities of the same logarithmic type as solutions for the entry of thin wedges and cones [1–3] in the neighbourhood of the apex, that is, of the type $-\varepsilon\ln r$, where r is the distance from the leading edge of the body, measured in a plane normal to it at some point. Thus, the non-uniformity domain [8] in which the outer (linear) solution of the problem φ_c becomes meaningless is of characteristic scale $r \sim e^{-1/\varepsilon}$. Cases in which the non-uniformity domain of the outer solution is not a "tube" of radius r surrounding the sharp leading edge but a "sphere" of the same radius about a certain point will be specified later.

To construct an inner solution, we change first to a Cartesian system of coordinates attached to the body. The appropriate form of Eq. (1.1) is

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=\left(1-M^2\right)\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}+2\frac{\upsilon_0}{c^2}\frac{\partial^2\varphi}{\partial t\partial x}+\frac{\dot{\upsilon}_0}{c^2}\frac{\partial\varphi}{\partial x} \tag{1.4}$$

$$x=x_1-\int_0^t \upsilon_0(t)dt,\quad y=y_1,\quad z=z_1,\quad M=\frac{\upsilon_0(t)}{c}$$

To simplify matters, let us assume that the sharp leading edge of the body (or one such edge) is a plane twice-differentiable curve in the plane $z = 0$, and that one of the two surfaces of the body which intersect in that edge are defined in its neighbourhood by the equation $\varepsilon f(x, y) - z = 0$. Thus, $f(x, y) = 0, z = 0$, is the equation of the leading edge.

We now change to an orthogonal curvilinear system of coordinates attached to the leading edge

$$n=f(x,y),\quad s=f_1(x,y),\quad z=z \tag{1.5}$$

where s may be, for example, the constant of the first integral of the equation $dy/dx = f_y/f_x$. Equation (1.4) and boundary condition (1.3) may be written for the selected part of the body surface as

$$\begin{gathered}\left[\left(1-M^2\right)f_x^2+f_y^2\right]\frac{\partial^2\varphi}{\partial n^2}+\left[\left(1-M^2\right)f_{1x}^2+f_{1y}^2\right]\frac{\partial^2\varphi}{\partial s^2}+\frac{\partial^2\varphi}{\partial z^2}-\\ -2M^2f_xf_{1x}\frac{\partial^2\varphi}{\partial n\partial s}+\left[\left(1-M^2\right)f_{xx}+f_{yy}\right]\frac{\partial\varphi}{\partial n}+\left[\left(\left(1-M^2\right)f_{1xx}+f_{1yy}\right]\frac{\partial\varphi}{\partial s}-\\ -\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}+2\frac{\upsilon_0}{c^2}\left(f_x\frac{\partial^2\varphi}{\partial t\partial n}+f_{1x}\frac{\partial^2\varphi}{\partial t\partial s}\right)+\frac{\dot{\upsilon}_0}{c^2}\left(f_x\frac{\partial\varphi}{\partial n}+f_{1x}\frac{\partial\varphi}{\partial s}\right)=0\end{gathered} \tag{1.6}$$

$$\varepsilon\left(f_x^2+f_y^2\right)\frac{\partial\varphi}{\partial n}-\frac{\partial\varphi}{\partial z}=\varepsilon f_x\upsilon_0(t) \tag{1.7}$$

In accordance with the above estimate for the scale of the neighbourhood of the leading edge where the outer (linear) solution becomes applicable, we introduce inner variables

$$n_i=ne^{1/\varepsilon},\quad z_i=ze^{1/\varepsilon} \tag{1.8}$$

Then, assuming the necessary restrictions on the orders of magnitude of the derivatives of $f(x, y)$ and $f_1(x, y)$ in the neighbourhood of the leading edges, we obtain the following relations for the principal terms (1.6) and (1.7)

$$\left[1 - M_n^2(t,s)\right]\Delta^2(s)\frac{\partial^2\varphi_i}{\partial n_i^2} + \frac{\partial^2\varphi_i}{\partial z_i^2} = 0 \tag{1.9}$$

$$\varepsilon\Delta^2(s)\frac{\partial\varphi_i}{\partial n_i} - \frac{\partial\varphi_i}{\partial z_i} = \varepsilon f_x(s)\upsilon_0(t)e^{-1/\varepsilon} \tag{1.10}$$

where $M_n(t, s) = v_0(t)|f_x(s)|/(c\Delta(s))$ is the Mach number of the velocity component normal to the leading edge at some points, $\Delta^2(s) = f_x^2 + f_y^2$ when $f(x, y) = 0$ and φ_i is the potential of the inner solution.

Since linear solutions of problems of thin three-dimensional bodies entering a fluid have logarithmic singularities in the neighbourhood of the leading edges at subsonic velocities of the latter ($M_n < 1$) [4, 5], it follows that (1.9) is an equation of the elliptic type. Introducing the new variables

$$\varphi_i = \frac{f_x}{\Delta^2}\upsilon_0 e^{-1/\varepsilon}n_i + \Phi_i, \quad n_i = \Delta m n_{i1} \tag{1.11}$$

$$\left(m = \sqrt{1 - M_n^2(t,s)}\right)$$

we obtain the following expressions for (1.9) and (1.10)

$$\frac{\partial^2\Phi_i}{\partial n_{i1}^2} + \frac{\partial^2\Phi_i}{\partial z_i^2} = 0 \tag{1.12}$$

$$\varepsilon\frac{\Delta}{m}\frac{\partial\Phi_i}{\partial n_{i1}} - \frac{\partial\Phi_i}{\partial z_i} = 0 \tag{1.13}$$

Thus, the inner problem has been reduced to solving a two-dimensional Laplace equation (1.12) for the potential Φ_i in the plane normal to the leading edge of the body at some point, subject to the Riemann–Hilbert condition (1.13) [9] on one side of the "wedge": $z_i = \varepsilon\Delta m n_{i1}$ (we have omitted terms of order ε^2 relative to unity), which is the image in the inner domain of the curve in which the surface $z = \varepsilon f(x, y)$ intersects the aforementioned normal plane. The variables t and s are the parameters of the inner problem. If the plane $z = 0$ is the plane of symmetry of the body in a small neighbourhood of the leading edge, one must assume, in addition to condition (1.13), the boundary condition $\partial\Phi_i/\partial z_i = 0$ for $z_i = 0, n_{i1} < 0$. In the most general case, when the surface that forms the leading edge together with the aforementioned intersection is defined in a neighbourhood of the edge by the equation $z = -\varepsilon_1 f(x, y)$, one must add a boundary condition on the second side of the "wedge" $z_i = -\varepsilon_1\Delta m n_{i1}$

$$\varepsilon_1\frac{\Delta}{m}\frac{\partial\Phi_i}{\partial n_{i1}} + \frac{\partial\Phi_i}{\partial z_i} = 0 \tag{1.14}$$

The solution of Eq. (1.12) is conveniently presented, as in [7], in polar coordinates

$$r_i = \sqrt{n_{i1}^2 + z_i^2}, \quad \operatorname{tg}\theta_i = z_i / n_{i1} \tag{1.15}$$

but in a form involving two arbitrary functions of the variables t and s, as well as an arbitrary parameter to satisfy boundary conditions (1.13) and (1.14)

$$\Phi_i = a(t,s)r_i^k\cos\left[k(\theta_i - \alpha)\right] + b(t,s) \tag{1.16}$$

Substituting (1.16) into (1.13) and (1.14) for $\theta_i = \varepsilon\Delta m$ and $\theta_i = 2\pi - \varepsilon_1\Delta m$, respectively, we obtain

$$k = 1 + \frac{(\varepsilon+\varepsilon_1)\Delta}{2\pi m}\left[1 + \frac{(\varepsilon+\varepsilon_1)\Delta m}{2\pi}\right], \quad \alpha = 2\pi\frac{\varepsilon(k-1)}{k(\varepsilon+\varepsilon_1)} \tag{1.17}$$

By (1.1), Φ_i is the potential of relative motion of the fluid for the inner problem; hence it is easy to find the streamline—the ray θ_{i0} of the polar system of coordinates r_i, θ_i—that enters the critical point on the leading edge. In the principal approximation, we have

$$\theta_{i0} = (\varepsilon - \varepsilon_1)\Delta/(2m) \tag{1.18}$$

Taking into account the formulae for changing to inner variables (1.4), (1.5), (1.8) and (1.11), we conclude that the position of the critical streamline (1.18) depends not only on the linear angles between the tangent planes to the surfaces whose intersection forms the leading edge at the point in question and the plane $z = 0$, but also on the Mach number of the velocity normal to the leading edge.

The functions $a(t, s)$ and $b(t, s)$ in (1.16) are found from the condition that the outer limit of the inner solution must match the inner limit of the outer solution [8]

$$[\varphi_i]_e = [\varphi_e]_i \tag{1.19}$$

In the problem being treated here these functions are more conveniently determined successively, first matching the velocity components of the inner and outer solutions, and then the potentials. After finding the functions $a(t, s)$ and $b(t, s)$, one constructs a uniformly valid composite solution of the problem using the formula

$$\varphi_c = \varphi_e - [\varphi_e]_i + \varphi_i \tag{1.20}$$

2. EXAMPLES AND REMARKS

Let us consider the problem of a cyclically-symmetric thin three-dimensional body with plane facets and an even number of cycles, entering a compressible fluid normal to its surface at a constant velocity v_0 [5]. The general linear solution of this problem is an angular superposition of solutions of a similar linear problem for a conical body with a thin rhomboid profile in cross-section (Fig. 1).

Figure 2 shows the existence domains of different entry conditions for a thin conical body with rhomboid profile, with different configurations of the acoustic waves, in the plane of the parameters $m = v_0/c$ and β. The dashed curves are described by the equations M tg$\beta = 1$ (curve l) and $M \sin\beta = 1$ (curve 2). Domain 1, situated below the straight line $M = 1$ and curve 1, corresponds to entirely subsonic entry of the body; domain 2, situated below the line $M = 1$ and above curve 1, corresponds to supersonic motion of the profile of the leading edge of the body along the free surface of the fluid; domain 3, bounded by the line $M = 1$ and by curves 1 and 2, corresponds to

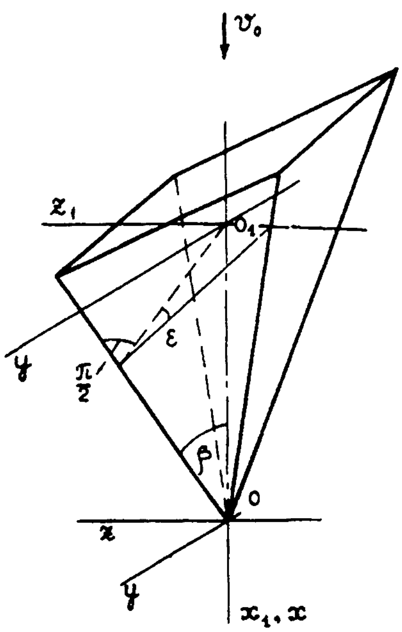

Fig. 1.

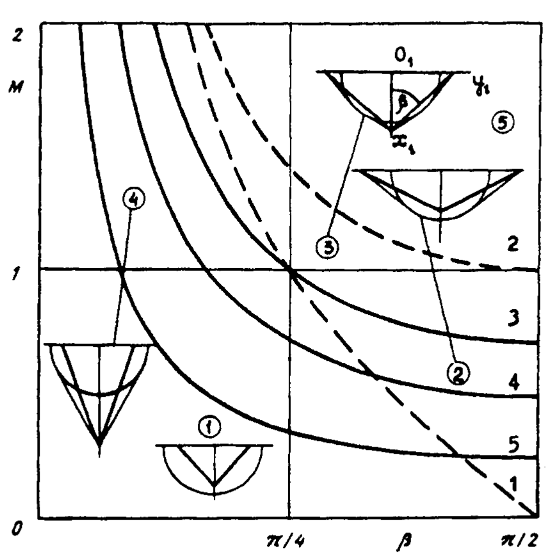

Fig. 2.

entirely subsonic motion of the leading edge ($M_h \equiv M \sin\beta < 1$) at supersonic velocities of the body nose and the profile of the leading edge along the free surface; domain 4, situated above the line $M = 1$ and below curve 1, corresponds to supersonic motion of the nose; domain 5, situated above curve 2, represents a supersonic mode of motion of the leading edge of the body ($M_n > 1$). The solid curves 3–5 correspond to constant velocities of the leading edges: $M_n = \sqrt{2}/2$, 0.5 and 0.25.

According to [5], the linear (outer) solutions of problems relating to the entry of the thin body that correspond to values of M and β in domains 1–4 (Fig. 2, below curve 2) involve logarithmic singularities at the leading edges.

Here are some illustrative constructions of uniformly valid solutions for regimes 2–4 in domains of conical motion of the disturbed fluid.

Let $M \operatorname{tg}\beta > 1$. The potential of the outer solution in the domain between a conical wave, with its vertex at a moving point of the intersection of the leading edge and the free surface of the liquid, a spherical wave and the free surface (Fig. 2, domains 2, 3) may be written as follows (see [5]):

$$\varphi_e = -\int_{t_1}^{t} F(x_1, y_1, z_1, t)\,dt, \quad t > t_1 \tag{2.1}$$

$$F(x_1, y_1, z_1, t) = \frac{\varepsilon v_0^2 h}{2\pi g}\sin\beta \ln\left|\frac{G_+}{G_-}\right|, \quad h = \frac{\operatorname{tg}\beta}{\sqrt{M^2\operatorname{tg}^2\beta - 1}}$$

$$G_\pm = \left[\sqrt{h^2(v_0 t - y_1\operatorname{ctg}\beta)^2 - (x_1^2 + z_1^2)} \pm g x_1\right]^2 + h^2 z_1^2$$

$$t_1 = \frac{\operatorname{ctg}\beta}{v_0}\left(y_1 + \sqrt{M^2\operatorname{tg}^2\beta - 1}\ \sqrt{x_1^2 + z_1^2}\right), \quad g = \frac{h}{\sin\beta}\sqrt{1 - M^2\sin^2\beta}$$

The facet of the body (Fig. 1) in the first quadrant of the x_1, y_1, z_1, system is described in variables x, y, z (1.4) by the function

$$z = -\varepsilon(x\sin\beta + y\cos\beta)$$

Then, according to Section 1, the leading edge of the body is defined by the equation $f(x, y) \equiv -(x\sin\beta + y\cos\beta) = 0$, and the variables n and s of the system of coordinates attached to the leading edge are defined by

$$n = -x_1\sin\beta - y_1\cos\beta + v_0 t\sin\beta$$
$$s = x_1\cos\beta - y_1\sin\beta + v_0 t\sin\beta\operatorname{tg}\beta \tag{2.2}$$

(the origin here is situated at the point of intersection of the leading edge and the free surface of the fluid), and the functions $M_n(t, s)$ and $\Delta(s)$ are constants

$$M_n = M\sin\beta, \quad \Delta = 1 \tag{2.3}$$

Omitting the laborious details, we will write the inner limits for the velocity components of the outer solution, projected onto the n, s, z axes, and the outer potential φ_e

$$[v_{ne}]_i = \frac{\varepsilon v_0}{\pi q_1}\left\{\ln\sqrt{\left(\frac{n}{s\cos\beta\sin\beta}\right)^2 + q_1^2\left(\frac{z}{s\cos\beta}\right)^2} - \ln(2q_1^2) - \sin^2\beta\ln q_2\right\} \tag{2.4}$$

$$[v_{se}]_i = \frac{\varepsilon v_0}{\pi q_1}\left\{\sin\beta\cos\beta\ \ln q_2 - \frac{n}{s}\right\} \tag{2.5}$$

$$[v_{ze}]_i = \frac{\varepsilon v_0\sin\beta}{\pi}\left\{\frac{\pi}{2} + \operatorname{arctg}\left(\frac{n}{mz}\right)\right\} \tag{2.6}$$

$$[\varphi_e]_i = n[v_{ne}]_i + s[v_{se}]_i + z[v_{ze}]_i \tag{2.7}$$

$$q_1 = g/h, \qquad q_2 = h^2(g + h)^{-2q_1}$$

As can be seen from (2.4)–(2.6), when the point approaches the leading edge at s = const, the velocity component v_{ne} involves a logarithmic singularity, as is the case in the perturbation of the pressure in the linear solution [5] (see (2.1), $\partial\varphi_e/\partial t$), the component v_{se} is continuous and v_{ze} depends on the direct of approach to the leading edge.

When the body profile is symmetric about the plane $z = 0$ (Fig. 1) and $\Delta = 1$ (see (2.3)), the parameters of the inner solution (1.17) become

$$k = 1 + \frac{\varepsilon}{\pi m}\left(1 + \frac{\varepsilon m}{\pi}\right), \quad \alpha = \pi\frac{k-1}{k} \tag{2.8}$$

and the potential of the inner solution (1.11) is

$$\varphi_i = -\upsilon_0 n \sin\beta - a(t,s) r_i^k \cos[k(\theta_i - \pi)] + b(t,s) \tag{2.9}$$

Taking (1.8), (1.11), (1.15) and (2.8) into consideration, we can write the potential of the inner solution, apart from terms of order ε^2 relative to unity, in the form

$$\begin{aligned}\varphi_i &= -\upsilon_0 n \sin\beta + \frac{a(t,s)}{q_1 \sin\beta}\exp\left(\frac{1}{\varepsilon} + \frac{1}{\pi m}\right) n\chi \times \\ &\times\left[1 + \frac{\varepsilon}{\pi^2} - \frac{\varepsilon}{\pi m}\ln q_1 + \frac{\varepsilon z}{\pi n}\left(\frac{\pi}{2} + \operatorname{arctg}\frac{n}{mz}\right)\right] + b(t,s) \\ \chi &= \left[(n/\sin\beta)^2 + q_1^2 z^2\right]^{\varepsilon/(2\pi m)}\end{aligned} \tag{2.10}$$

To determine the function $a(t, s)$, we find the outer limit $[v_{ni}]_e$ of the velocity component v_{ni} of the inner solution. Expressing $a(t, s)$ as a series in powers of ε: $a(t, s) = a_0 + \varepsilon a_1 + \ldots$, and equating $[v_{ni}]_e$ and $[v_{ne}]_i$ (see (2.4)), we obtain

$$\begin{aligned}a_0 &= \upsilon_0 m \sin\beta \exp\left(-\frac{1}{\varepsilon} - \frac{1}{\pi m}\right) \\ a_1 &= -\frac{a_0}{\pi m}\left\{1 + \frac{m}{\pi} + \ln|2q_1 s\cos\beta| + \sin^2\beta \ln q_2\right\}\end{aligned} \tag{2.11}$$

Using (2.11) to determine the outer limit of the potential of the inner solution (2.10), we deduce from (1.19) that

$$b(t,s) = \varepsilon\frac{\upsilon_0 \sin\beta\cos\beta}{\pi q_1} s \ln q_2 \tag{2.12}$$

Finally, the expression for the potential of the inner solution is

$$\begin{aligned}\varphi_i &= \upsilon_0 n \sin\beta\left\{\left[\left(\frac{n}{s\cos\beta\sin\beta}\right)^2 + q_1^2\left(\frac{z}{s\cos\beta}\right)^2\right]^{\varepsilon/(2\pi m)} \times\right. \\ &\left.\times\left[1 + \frac{\varepsilon}{\pi m}\left(m\frac{z}{n}\left(\frac{\pi}{2} + \operatorname{arctg}\frac{n}{mz}\right) - 1 - \sin^2\beta\ln q_2\right)\right] - 1\right\} + \varepsilon\frac{\upsilon_0 \sin\beta\cos\beta}{\pi q_1} s\ln q_2\end{aligned} \tag{2.13}$$

It can be verified that $[v_{si}]_e$ and $[v_{zi}]_e$ are identical with (2.5) and (2.6), respectively.

The composite solution for the potential φ_c is determined from formula (1.20), using (2.1), (2.7) and (2.13). Analysis of that solution shows that the velocity components of the uniformly valid solution are continuous at the leading edge.

Let us determine the pressure at the leading edge of a thin conical body with rhomboid profile (Fig. 1) when $M \operatorname{tg}\beta > 1$, in the domain of conical flow (Fig. 2, domains 2 and 3). We will use the Cauchy–Lagrange integral (1.2), with φ replaced by φ_c (1.20). Omitting the algebra, we find the reduced pressure coefficient at $n = z = 0$, calculating the velocity head q for a velocity normal to the leading edge

$$\begin{aligned}\bar{c}_{p0} &\equiv \frac{\pi}{4\varepsilon}(c_{p0} - 1) = \sin\beta\ln\left[(1+q_1)\left(\sqrt{1-q_1^2}\right)^{1/q_1 - 1}\right] \\ c_{p0} &= (p - p_0)\big|_{n,z=0} q^{-1}, \quad q = \frac{1}{2}\rho_0 \upsilon_0^2 \sin^2\beta\end{aligned} \tag{2.14}$$

As $\beta \to \pi/2$ the solution of the linear problem for a thin body with rhomboid profile entering a compressible fluid tends to the corresponding solution of the plane problem—entry of a thin wedge [5]. As analysis of formula (2.14) shows, the expression for the pressure at the edge obtained from the Cauchy–Lagrange integral using the uniformly valid solution is identical, as $\beta \to \pi/2$ for small M values, with the corresponding formula for the pressure at the wedge apex [6].

Omitting the reasoning and the algebra, we write the expression for the reduced pressure coefficient at the edge of a thin body with rhomboid profile in the conical flow domain, valid in a neighbourhood of the body nose for entry conditions $M > 1$ (Fig. 2, domains 3 and 4)

$$\tilde{c}_{p0} = -\frac{\cos^2\beta}{m}\ln\left(\mathrm{tg}\beta\sqrt{M^2-1}\right) \tag{2.15}$$

The question arises as to whether a uniformly valid solution constructed for a single thin body (Fig. 1) holds for a cyclically-symmetric body (CSB); in particular, how does one determine the pressure at the leading edge?

Obviously, in an entry regime M tg$\beta > 1$, there will always be a domain of the governing parameters and a finite range of s values in which the conical flow domain realized in the neighbourhood of a point where the leading edge of one of the constituent cycles of the body intersects the free surface of the fluid is not affected by the other cycles.

Consequently, a uniformly valid solution constructed for that domain in the entry problem for a single thin body (Figs 1 and 2, domains 2 and 3) may be used for a CSB also.

In other flow domains, however, containing a subsonic leading edge ($M_n < 1$), the construction of a uniformly valid solution must allow for the combined influence of the constituent cycles. This is particularly true in the conical domain of flow around the nose of the body at $M > 1$ (Fig. 2, domains 3 and 4), and thus formula (2.15) must be corrected taking an appropriate number of constituent cycles of the body into account.

We shall consider the main features involved in constructing a uniformly valid solution for the entry of a thin CSB. Now, the outer (linear) solution in the neighbourhood of any of the leading edges is a superposition of the main solution generated by the cycle to which that edge belongs, which involves a logarithmic singularity at the edge, and of the effects of the other cycles, which do not contribute singularities to the general solution in the neighbourhood of the edge. It follows that, for example, the inner asymptotic behaviour of the outer solution for the velocity component v_n: $[v^c_{ne}]_i$ may be written as

$$[v^c_{ne}]_i = \left[v^0_{ne}\right]_i + \varepsilon c(t,s) + O(\varepsilon e^{-1/\varepsilon}) \tag{2.16}$$

where $[v^0_{ne}]_i$ is the inner limit of the outer solution for the corresponding velocity component in the main linear problem, corresponding to entry of a single cycle (for example, that illustrated in Fig. 1) and $c(t, s)$ is a function describing the effect of the other cycles.

According to (2.16), the formula for the potential in the neighbourhood of the leading edge in the general case will be

$$[\varphi^c_e]_i = \left[\varphi^0_e\right]_i + \varepsilon c(t,s)n + \varepsilon c_1(t,s)z + \varepsilon d(t,s) + O(\varepsilon e^{-2/\varepsilon}) \tag{2.17}$$

where $c_1(t, s)$ and $d(t, s)$ are certain functions that describe the effect of the other cycles. Because of the cyclic symmetry of the body, $c_1(t, s) = 0$, since disturbances arriving at the plane $z = 0$ from the other cycles cancel out, so that $v^c_{ze}(z = 0) - v^{0\prime}_{ze}(z = 0) = 0$. Accordingly, analysis of a formulae (1.19), (2.9) and (2.17) shows that the function $c(t, s)$ occurs in the coefficient a_1 (2.11) of the series for the function $a(t, s)$, and the function $d(t, s)$ similarly for the function $b(t, s)$ (2.9). We may therefore conclude, in view of (1.2), (1.20), (2.8), (2.9) and (2.17), that the pressure at the edge will depend, apart from terms of the order of ε^2, only on the derivative of $d(t, s)$ with respect to time, which occurs in the formula additively, i.e. in the same way as in the linear theory for evaluating the pressure using the linearized Cauchy–Lagrange integral.

Thus, in order to determine the pressure at the leading edge of the CSB in domains where the cycles influence one another, it will suffice to sum up the pressure at the edge evaluated in the principal problem, using the uniformly valid solution and the non-linear Cauchy–Lagrange integral, and the pressure perturbations contributed by the other cycles at the point, evaluated using the linear solution.

Figure 3 shows the results of computations of the normalized reduced pressure coefficient $c_p = \tilde{c}_{p0}/\ln 2$ using formulae (2.14) (the solid curves) and (2.15) (the dashed curves). This normalization is necessary because, in the first case, corresponding to the pressure at the edge in the conical flow domain when M tg$\beta > 1$, we have $\tilde{c}_{p0} = \ln 2$ as $\beta \to \pi/2$ and $M \to 0$. The numbers on the curves correspond to the values of $10 \times M_n$.

The dash–dot curve A limiting the solid curves on the left is the image of curve 1 in Fig. 2 (M tg$\beta = 1$). The dash–dot curve B demarcates domains representing subsonic motion of the body ($M < 1$, to the right of the curve) and supersonic motion. When $M_n > \sqrt{2}/2$ (see Fig. 2, domain 3) the two domains of conical flow considered previously occur in the neighbourhood of the leading edge. In these motions, as we have calculated, the pressure at the edge in the neighbourhood of the nose exceeds the pressure in the neighbourhood of the free surface of the fluid (Fig. 3, compare the ordinates of the dashed and solid curves 8 and 9 for identical values of the angle β).

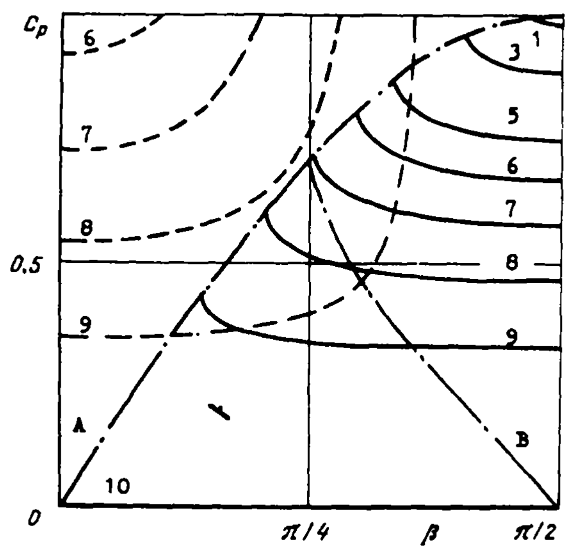

Fig. 3.

It should be noted, however, that the solution constructed for $M > 1$ in the conical flow domain at the nose becomes inapplicable as $M \to 1$. This is indicated by the logarithmic singularity in formula (2.15) (see also the dashed curves in Fig. 3). Although the body entering the fluid is by assumption thin, the angles of deflection of the flow are finite, and when $M \gtrsim 1$ the wave must become detached from the tip. This indicates that the acoustic model is no longer valid in this limiting case.

A similar situation occurs in the solution for the conical flow domain in the neighbourhood of the free surface of the fluid. Although the theory gives finite values for the pressure at the leading edge when $M \operatorname{tg} \beta > 0$ (2.14) (Fig. 3, the solid curves in the neighbourhood of the dash–dot curve A), the acoustical model is inapplicable here also. In other words, both in the first case and in the second, given ε, the conical flow domains, bounded by spherical and conical waves (Fig. 2, domains 2–4), in which the solutions we have constructed are usable are limited in size. In addition, the solution for a conical flow domain in the case $M \operatorname{tg} \beta > 1$ cannot be extended to the whole leading edge from the spherical wave to the free surface (Fig. 2, domains 2 and 3). As $s \to 0$ ($s = O(e^{-1/\varepsilon})$) the points on the leading edge fall within the neighbourhood of the curve in which the body surface intersects the free surface of the liquid, where the solution must be constructed taking the shape of the free surface into account.

To conclude, we observe that the technique evolved here for constructing uniformly valid solutions for the problem of a thin three-dimensional body entering a fluid (Section 1) presupposes that the leading edges are sufficiently smooth. In particular, it is not applicable when the leading edge has a sharp bend. Thus, in subsonic motion (Fig. 2, domain 1), the characteristics of the linear (outer) solution of the problem have a logarithmic singularity at the body nose when it is approached by a point of the field of disturbed flow in any direction. This indicates that the non-uniformity domain of the outer solution is not in this case a "tube", as it is at the leading edge, but a "sphere" of characteristic radius $r = O(e^{-1/\varepsilon})$. Therefore, the inner variables in this case must be introduced for all three Cartesian coordinates x, y, z (1.4), (1.8), which leads to an inner problem for a three-dimensional Laplace equation with appropriate boundary conditions at the body surface in the neighbourhood of the nose.

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